* Volume :

Volume
- slicing
  - base
    - Semicircle: $A = \frac{1}{2}\pi r^2$
    - Square: $A = (s \cdot d)^2$
    - triangle: $A = \frac{1}{2} \cdot b \cdot h$
- rotation
  - Disc
  - washer
  - cylindrical shell

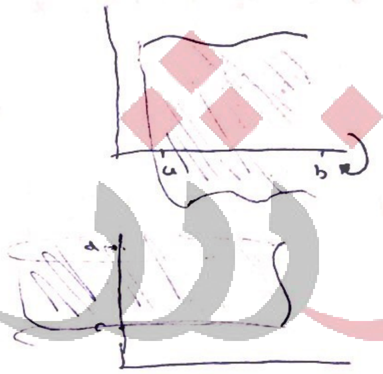
$\underbrace{\text{Disc} \quad \text{washer}}_{\text{base is circle}}$

* Disc : when rotating about x-axis :

$$V = \pi \int_a^b (f(x))^2 \, dx$$

* Disc : when rotating about y-axis :

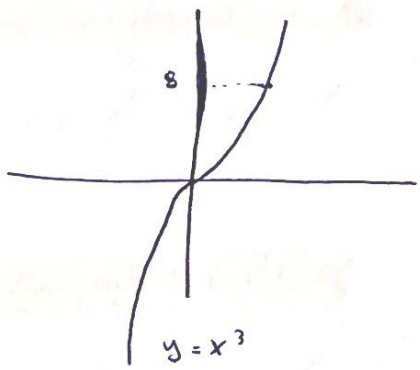
$$V = \pi \int_c^d (f(y))^2 \, dy$$



*** Example :** find the volume obtained by revolving the region b-dd by $y = x^3$, $x = 0$, $y = 8$ about y-axis

$x = 0$, $y = 0$

$$V = \pi \int_0^8 (\sqrt[3]{y})^2 \cdot dy = \pi y^{\frac{5}{3}} * \frac{3}{5} \Big|_0^8 =$$



* washer
  - when rotating about x-axis

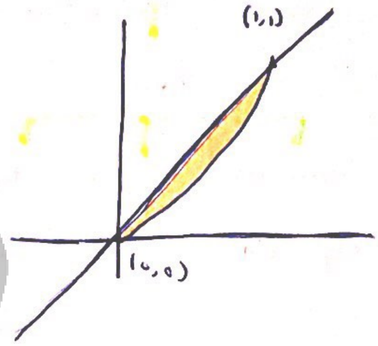
$$V = \pi \int_a^b \left(f(x)^2 - g(x)^2 \right) dx$$

  - when rotating about y-axis

$$V = \pi \int_a^b \left(f(y)^2 - g(y)^2 \right) dy$$

*** Example:** find the volume obtained by rotating the region bounded by $y = x^2$ and $x = y$ about:

1) x-axis
2) y-axis



* finding the interaction points.

$$x = x^2$$
$$0 = x^2 - x \rightsquigarrow x(x-1) = 0$$
$$x = 0 \;,\; 1$$
$$y = 0 \;,\; 1$$

1) $$V = \pi \int_0^1 x^2 - x^4 \, dx = \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5} \right)$$

2) $$V = \pi \int_0^1 (\sqrt{y})^2 - y^2 \, dy = \pi \left(\frac{y^2}{2} - \frac{y^3}{3} \right) \Big|_0^1 = \pi \left(\frac{1}{2} - \frac{1}{3} \right)$$

* * * * * * *

· **Method of cylindrical shell:**

* **if rotating about x-axis:**

$$V = 2\pi \int_c^d (f(y) - g(y)) \cdot y \, dy$$

⇝ distance between x-axis

* **if rotating about $y = -a$:**

$$V = 2\pi \int_c^a (y+a)(f(y) - g(y)) \, dy$$

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* **if Rotating about y axis**

$$V = 2\pi \int_a^b f(x) - g(x) \cdot x \, dx$$

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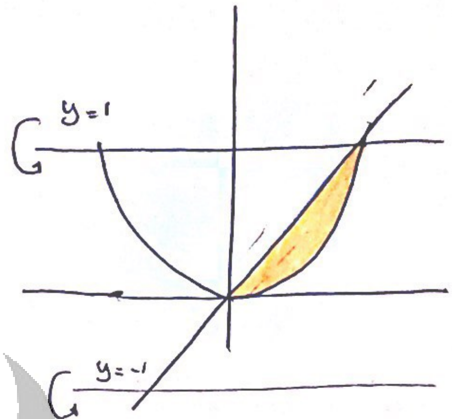
* **if Rotating about $x = -c$**

$$V = 2\pi \int_c^d (f(x) - g(x))(x + c) \, dx$$

* Example : Using cylindrical shell

1) $V = 2\pi \int_0^1 (\sqrt{y} - y)(1-y)\, dy$

2) $V = 2\pi \int_0^1 (\sqrt{y} - y)(y+1)\, dy$



- Arclength :

$$L = \int_a^c \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx .$$

$$L = \int_a^c \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy .$$

اثبات :

$$L^2 = (dx)^2 + (dy)^2$$

$$\frac{L^2}{dx^2} = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\frac{L}{dx} = \sqrt{1 + f'(x)^2}$$

$$L = \sqrt{1 + f'(x)^2}\, dx$$

* Example : find the arc length for $y = x^2 - \frac{1}{2}\ln x$ $(1,1)$ $(e, e^2 - \frac{1}{8})$

$$\frac{dy}{dx} = 2x - \frac{1}{2x} \qquad \frac{4x^2 - 1}{2x}$$

$$\left(\frac{dy}{dx}\right)^2 = \cancel{4x^2 - \frac{1}{2} + \frac{1}{4x^2}} \qquad \frac{16x^4 - 8x^2 + 1}{4x^2}$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{16x^4 - 8x^2 + 1}{4x^2}\right)}$$

$$\int_1^e \quad dx$$

* chapter 11: Sequences and series.

$$\left[a_n\right]_{n=1}^{\infty} = a_1, a_2, a_3, a_4, a_5 \dots \quad \text{infinite sequence}$$

a_1 → first term, a_2 → Second term

a_1, a_3, a_5 → odd terms
a_2, a_4, a_6 → even terms

---

* $2, 4, 6, 8, \dots \quad \left[2n\right]_{n=1}^{\infty}$
* $1, 3, 5, 7, \dots \quad \left[2n-1\right]_{n=1}^{\infty}$
* $2, 4, 8, \dots \quad \left[2^n\right]_{n=1}^{\infty}$
* $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \quad \left[\frac{n}{n+1}\right]_{n=1}^{\infty}$
* $1, -1, 1, -1$

$(-1)^{n-1} \quad (-1)^{n+1} \quad (-1)^n$

* * * * * * *

- Monotonic Sequence:

A seq $\left[a_n\right]_{n=*}^{\infty}$ → is increasing $a_n' > 0$; decreasing $a_n < 0$

* A sequence is said to be converge if $\lim_{n\to\infty} a = \text{exist}$ otherwise its diverge.

* Even and odd terms should converge to same limit

- Example:

$$\left[\frac{2n}{3n+1}\right]_{n=7}^{\infty} \rightsquigarrow \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3} \quad \text{converge}$$

$$\left[\frac{n^2}{n+1} \quad \cancel{\frac{n+1}{n^2+5n}}\right]_{n=1}^{\infty} \rightsquigarrow \lim_{n\to\infty} \frac{n^2}{n+1} = \infty \quad \text{diverge}$$

$$\left[\frac{n+1}{n^2+5n}\right]_{n=1}^{\infty} \rightsquigarrow \lim_{n\to\infty} \frac{n+1}{n^2+5n} = 0 \quad \text{converge}$$

$\frac{\text{poly}}{\text{poly}}$

درجة البسط = المقام ←
$\frac{\text{أكبر قيمة}}{\text{أكبر قيمة}}$

درجة البسط > مقام ←
$-\infty, \infty$

درجة البسط < مقام ←
0

$$\left[\sqrt[n]{n}\right]_{n=1}^{\infty} \qquad \lim_{n\to\infty} n^{1/n} = \infty^{0} \;!!$$

$$y = n^{1/n} \rightarrow \ln y = \frac{1}{n}\ln n$$

$$\lim_{n\to\infty} \frac{\ln n}{n} \xrightarrow{\text{lopetal}} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \frac{1}{\infty} = 0 \text{ converge}$$

$$\lim_{n\to\infty} n^{1/n} = e^{0} = 1$$

* * * * * *

*** Rule ; $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{n} = e^{a}$**

- $\lim_{n\to\infty}\left(1-\frac{2}{n}\right)^{n} = e^{-2}$ converge

- $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{3n} = \left(\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n}\right)^{3} = (e^{2})^{3} = e^{6}$ converge

- $\left[e^{-2n} \cdot \cos n\right]_{n=1}^{\infty} \qquad \lim_{n\to\infty} e^{-2n}\cdot \cos n = e^{-\infty} \cdot \cos\infty$

  $0 \;.\;$ doesn't exist

$$-1 \leq \cos n \leq 1$$

$$-e^{-2n} \leq e^{-2n}\cos n \leq e^{-2n}$$

limit = 0   limit = 0

* by squeezing theorem $\lim_{n\to\infty} e^{-2n}\cdot\cos n = 0$

Converg

- $\left[(-1)^{n}\right]_{n=1}^{\infty}$
  - odd terms : $-1, -1, -1, \dots \xrightarrow{\infty} -1$
  - even terms : $1, 1, 1, 1 \xrightarrow{\infty} 1$

$\lim_{n\to\infty} (-1)^{n} = \pm 1$ diverge

* they didn't converge to same limit.

- $\left[(-1)^{n}\frac{1}{n}\right]_{n=1}^{\infty} \qquad \lim_{n\to\infty}\frac{1}{n} \;*\; \lim_{n\to\infty}(-1)^{n}$

  $0 \;,\; \pm 1 = 0$ converge

* odd and even terms converged to same limit

- $\left[\left(\frac{n}{n+1}\right)^n\right]_{n=1}^{\infty} \rightsquigarrow \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n}$

$$\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)^{-1} = e^{-1} \text{ converge}$$

---

* Recursive Sequence :

find the limit of the sequence such that the limit exist ?

$\sqrt{2}\ ,\ \sqrt{2+\sqrt{2}}\ ,\ \sqrt{2+\sqrt{2+\sqrt{2}}}\ ,\ \ldots\ldots$

$a_1 \qquad a_2 = \sqrt{a_1+2} \qquad a_3 = \sqrt{2+a_2}$

$a_{n+1} = \sqrt{2+a_n} \qquad \lim_{n\to\infty} a_n = L \qquad \lim_{n\to\infty} a_{n+1} = L$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\sqrt{2+a_n} = \sqrt{2+\lim_{n\to\infty} a_n}$$

$$L = \sqrt{2+L} \rightsquigarrow L^2 - L - 2 = 0 \qquad L = 2,\ \cancel{-1}$$

* * * * * *

- Infinite series : $[a_n]_{n=1}^{\infty}$ be infinite Seq.

$S_1 = a_1$ = first partial sum

$S_2 = a_1 + a_2$ = second partial sum .

$S_3 = a_1 + a_2 + a_3$ = third partial sum .

$\vdots \qquad \vdots$

$S_n = a_1 + a_2 + a_3 + \ldots + a_n$ = n-th partial sum

$$S_n = \sum_{i=1}^{n} a_i$$

$\rightsquigarrow$ said to be converge if limit exist

*** Example:** is the following series conv or div :

$$\sum_{k=1}^{\infty} \tan^{-1}k - \tan^{-1}(k+1)$$

$$S_n = (\tan^{-1}1 - \cancel{\tan^{-1}2}) + (\cancel{\tan^{-1}2} - \cancel{\tan^{-1}3}) + (\cancel{\tan^{-1}3} - \cancel{\tan^{-1}4}) + \ldots\ldots\ldots + (\cancel{\tan^{-1}(n-1)} - \cancel{\tan^{-1}n}) +$$

$$(\cancel{\tan^{-1}n} - \tan^{-1}(n+1)) = \tan^{-1}(1) - \tan^{-1}(n+1)$$

$$\frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4} \text{ converge}$$

* **Example**: $\sum_{k=2}^{\infty} \frac{1}{k^2+k} = \sum_{k=2}^{n} \frac{1}{k} - \frac{1}{k+1}$

$$= \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \dots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$\frac{1}{2} - \frac{1}{n+1}$ if we took limit

$\lim_{n\to\infty} \frac{1}{2} - \lim_{n\to\infty} \frac{1}{n+1} = \frac{1}{2}$ converg ✓

$$\left.\begin{aligned} &* \sum_{n=1}^{\infty} a_n \pm b_n = \sum a_n \pm \sum b_n \\ &* \sum_{n=1}^{\infty} c\,a_n = C \sum_{n=1}^{\infty} a_n \end{aligned}\right\} \text{properties}$$

* * * * * *

- **Geometric series**: $\sum_{n=0}^{\infty} a(r^n) = a + ar + ar^2 + \dots$

كل Term مقسوم على اللي قبله Constant Geometric

a = constant

r = base

$\frac{ar}{a} = r \qquad \frac{ar^2}{ar} = r$

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \text{conv if } |r| < 1 \rightsquigarrow \frac{a}{1-r} & \text{بشرط } n=0 \\ \text{div if } |r| \geq 1 \end{cases}$$

Question: Find the sum: $\sum_{k=2}^{\infty} \ln\left(1 - \frac{1}{k^2}\right)$

$$S_n = \sum_{k=2}^{n} \ln\left(\frac{k^2-1}{k^2}\right) = \sum_{k=2}^{\infty} \ln(k^2-1) - \ln k^2$$

$$\sum_{k=2}^{n} \ln(k-1) + \ln(k+1) - 2\ln k \rightsquigarrow -\ln k - \ln k$$

$$S_n = (\cancel{\ln 1} + \cancel{\ln 3} - 2\ln 2) + (\cancel{\ln 2} + \cancel{\ln 4} - \cancel{\ln 3} - \cancel{\ln 3}) + (\cancel{\ln 3} + \cancel{\ln 5} - \cancel{\ln 4} - \cancel{\ln 4})$$
$$+ (\cancel{\ln 4} + \cancel{\ln 6} - \cancel{\ln 5} - \cancel{\ln 5}) + (\cancel{\ln 5} + \ln 7 - \cancel{\ln 6} - \ln 6) + (\ln(n-1) + \ln(n+1) - \ln n - \ln n)$$

$$S_n = \ln 1 - \ln 2 + \ln(n+1) - \ln n =$$

* Examples:

1) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$

$r=1/2 \quad a=\frac{1}{2} \quad r<1$ then conv $\rightsquigarrow$ sum of $=\frac{1/2}{1-1/2}=1$

2) $1+2+4+8+\cdots$

$a=1$ then diverge because $|r|>1$

$r=2$

3) $\sum_{n=0}^{\infty} 2^{2n}\cdot 3^{1-n} = \sum_{n=0}^{\infty} 4^n\cdot\frac{3}{3^n} = \sum_{n=0}^{\infty} 3\cdot\left(\frac{4}{3}\right)^n$

$3+4+\frac{16}{3}+\frac{64}{27}$ $\quad r=4/3>1 \rightsquigarrow$ diverge

4) $\sum_{n=1}^{\infty}(e/\pi)^{n-1} = \sum_{n=1}^{\infty}(e/\pi)^{-1}(e/\pi)^n = \sum_{n=0}^{\infty}\frac{\pi}{e}*\left(\frac{e}{\pi}\right)^{n+1}$

$=\sum_{n=0}^{\infty}\cancel{\frac{\pi}{e}}*\cancel{\frac{e}{\pi}}*\left(\frac{e}{\pi}\right)^n = \frac{2.7}{3.14}<1 \rightsquigarrow$ converge

5) $0.3\overline{17} = 0.3+0.017+0.00017+0.0000017+\cdots$

$r=\frac{1}{100}, \quad a=0.017 \quad \rightsquigarrow$ converge $\frac{0.017}{1-0.01}=\frac{17*10^{-3}}{990*10^{-3}}=\frac{17}{990}$

$\frac{3}{10}+\frac{17}{990}=\frac{314}{990}=0.31717$

6) $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}+\left(\frac{1}{2}\right)^{2k+1} =$

$\underbrace{\sum_{k=1}^{\infty}\frac{1}{k(k+1)}}_{\lim_{n\to\infty} S_n} + \underbrace{\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{2k+1}}_{\text{geometric}} \qquad S_{total}=S_1+S_2$

* Divergent test:

- if $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=*}^{\infty} a_n$ is divergent
- if $\lim_{n\to\infty} a_n = 0 \Rightarrow$ test fail

1) $\sum_{n=1}^{\infty} \frac{2n+1}{7-3n} \rightsquigarrow \lim \frac{2n+1}{7-3n} = \frac{-2}{3} \neq 0$ div

2) $\sum_{n=5}^{\infty} \left(1+\frac{2}{n}\right)^{3n} \rightsquigarrow \left(\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n\right)^3 = (e^2)^3 = e^6 \neq 0$ diverge

* Integral test:

given $\sum_{n=1}^{\infty} a_n$ if
- a_n is positive term (+ve)
- a_n is continuous on $[b, \infty)$
- a_n' is < 0 (decreasing)

* rule: p-series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ conv if $p > 1$

div if $p \leq 1$

* Examples:

$\sum_{n=1}^{\infty} \frac{1}{n^2}$

positive term ✓

continuose on $[1, \infty)$ ✓

$a_n' = \frac{-2}{n^3} < 0$ ✓

$\int_1^{\infty} \frac{1}{x^2}$ $p = 2 > 1 \rightsquigarrow$ converge

* Example:

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$a_n = (+ve)$ ✓

continueouse ✓

$a'_n = \frac{-2}{(n^2+1)^2} < 0$ ✓

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{L\to\infty} \int_1^{L} \frac{1}{x^2+1}\,dx = \lim_{L\to\infty} \tan^{-1} x \Big|_1^L$$

$$= \lim_{L\to\infty} \tan^{-1}\infty - \tan^{-1} 1$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \text{ converge}$$

*** Example:** $\sum_{n=3}^{\infty} \frac{\ln n}{n}$

a_n = positive terms

$a_n' = \frac{n\frac{1}{n} - \ln n}{n^2} < 0$

a is continueose

$$\int_3^{\infty} \frac{\ln x}{x}\,dx = \lim_{L \to \infty} \int_3^{L} \frac{\ln x}{x}\,dx$$

$\ln x = z$

$\frac{1}{\cancel{x}}dx = dz$

$dx = x\,dz$

$$= \lim_{L \to \infty} \int_3^{L} z\,dz = \frac{z^2}{2}\Big|_3^{L}$$

$$= \frac{(\ln x)^2}{2}\Big|_3^{L} = \frac{(\ln L)^2}{2} - \frac{(\ln 3)^2}{2} = \infty \text{ diverge}$$

* * * * * * *

* Comparison theorem (C.T):

Given $\sum_{n}^{\infty} a_n$ $a_n \geq 0$ (+ve terms)

- Obtain $\sum_{n}^{\infty} b_n$, $b_n \geq 0$ → if $b_n \leq a_n$ and $\sum_{n}^{\infty} b_n$ is div ⟹ $\sum_{n}^{\infty} a_n$ is divergant

→ if $a_n \leq b_n$ and $\sum_{n}^{\infty} b_n$ is conv ⟹ $\sum_{n}^{\infty} a_n$ is conv

*** otherwise test fails**

*** Example:** $\sum_{n=1}^{\infty} \frac{1}{3n^2+n+4} \geq 0$

- $3n^2+n+4 > n^2$

- $\frac{1}{3n^2+n+4} < \frac{1}{n^2} \rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ $p = 2 > 1$ Convergant

$\therefore \sum_{n=1}^{\infty} \frac{1}{3n^2+n+4}$ conv. by Comparison theorem.

* limit comparison (L. Comparison Test)

given $\sum^{\infty} a_n$, $a_n \geq 0$

obtain $\sum b_n$, $b_n \geq 0$ and take $\lim_{n \to \infty} \frac{a_n}{b_n} = L$

* if $\rightsquigarrow$ $L > 0$ and $\sum b_n$ is conv $\rightsquigarrow$ $\sum a_n$ is conv.
  $\sum b_n$ is div $\rightsquigarrow$ $\sum a_n$ is div.

* if $\rightsquigarrow$ $L = 0$ and $\sum b_n$ is conv $\rightsquigarrow$ $\sum a_n$ is Conv.

* if $\rightsquigarrow$ $L = \infty$ and $\sum b_n$ is div $\rightsquigarrow$ $\sum a_n$ is div

i) Test for Convergance:

* $\sum_{n=1}^{\infty} \frac{3n^2 - 2n + 4}{n^5 - n^3 + 2}$ $\rightsquigarrow$ $a_n \geq 0$

$b_n = \frac{n^2}{n^5} = \frac{1}{n^3}$

$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left(\frac{3n^2 - 2n + 4}{n^5 - n^3 + 2} \right) * \frac{n^3}{1} = 3$ $3 > 0$ case 1

$\sum b_n = \sum \frac{1}{n^3}$ $\rightsquigarrow$ p $3 > 1$ ∴ conv. then $\sum a_n$: conv.

$\sum a_n$ Conv by L.C.T.

* $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n} + 5}$ $a_n \geq 0$ $b_n = \frac{1}{\sqrt[3]{n}}$

$\lim \frac{1}{\sqrt[3]{n} + 5} * \sqrt[3]{n} = 1 > 0$ case 1

$2^n + 1 > 2^n$

* $\sum_{n=0}^{\infty} \frac{1}{2^n + 1}$ $\rightsquigarrow$ $\frac{1}{2^n + 1} < \frac{1}{2^n}$